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## ANGULAR MOMENTUM DECOMPOSITION OF CHIRAL MULTIPLETS IN FRONT FORM

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In this article we derive the unitary transformation that relates the  $q\bar{q}$  chiral basis  $\{R; IJ^{PC}\}$  to the  $\{I; {}^{2S+1}L_J\}$ -basis in a front-form framework. From the most general expression for the Clebsch-Gordan coefficients of the Poincaré group one can see that the chiral limit brings the angular momentum coupling into a simple form that permits the relation in terms of  $SU(2)$  Clebsch-Gordan coefficients. We demonstrate that such a transformation is identical to the one was obtained for canonical spin in the instant form.

*Keywords:* Front form; Poincaré invariance; Chiral multiplets.

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### 1. Motivation

It has been shown in Ref. 1 that there is a unitary transformation that relates the  $q\bar{q}$  chiral basis, usually represented as  $\{R; IJ^{PC}\}$ , where  $R$  is the index of the chiral representation ( $R = (0, 0)$ ,  $(1/2, 1/2)_a$ ,  $(1/2, 1/2)_b$ , or  $(0, 1) + (1, 0)$ ),  $I$  is the quantum number of isospin, and  $J^{PC}$  indicates the total angular momentum of the state with definite parity and charge, and the  $\{I; {}^{2S+1}L_J\}$  basis, which regards the spin-orbit angular momentum coupling used in nonrelativistic quantum mechanics. This allows one to write a particular state belonging to a chiral multiplet with quantum numbers  $J^{PC}$ , as a superposition of states of the nonrelativistic-inspired  $\{I; {}^{2S+1}L_J\}$  classification scheme.

A chiral state with definite parity  $|R; IJ^{PC}\rangle$  can be decomposed as a superposition of helicity states without definite parity  $|J\lambda_1\lambda_2\rangle$  through<sup>1,2</sup>

$$|R; IJ^{PC}\rangle = \sum_{\lambda_1\lambda_2} \sum_{i_1i_2} \chi_{\lambda_1\lambda_2}^{RPI} C_{(1/2)i_1(1/2)i_2}^{Ii} |i_1\rangle |i_2\rangle |J\lambda_1\lambda_2\rangle \quad (1)$$

where  $i_{1(2)}$  and  $\lambda_{1(2)}$  are individual isospin and helicity respectively. Coefficients  $\chi_{\lambda_1\lambda_2}^{RPI}$  relate the helicity basis to the chiral basis with definite parity in the state. They can be found in Refs. 1, 2.  $C_{s_1\sigma_1 s_2\sigma_2}^{JM}$  are the usual  $SU(2)$  Clebsch-Gordan coefficients.

2 *María Gómez-Rocha*

Two-particle helicity states  $|J\lambda_1\lambda_2\rangle$  can be written in terms of vectors in the  $\{I; {}^{2S+1}L_J\}$  basis once one knows the expression for the matrix elements<sup>3</sup>

$$\langle J\lambda_1\lambda_2 | {}^{2S+1}L_J \rangle = \sqrt{\frac{2L+1}{2J+1}} C_{(1/2)\lambda_1(1/2)-\lambda_2}^{S\Lambda} C_{L0S\Lambda}^{J\Lambda} \quad (2)$$

It represents the angular momentum coupling of a two-particle state with individual helicities  $\lambda_1, \lambda_2$  (with  $\Lambda = \lambda_1 - \lambda_2$ ) to a system of total spin  $S$  and and orbital angular momentum  $L$ .

Combining (1) and (2) one finds

$$\begin{aligned} |R; IJ^{PC}\rangle &= \sum_{LS} \sum_{\lambda_1\lambda_2} \sum_{i_1i_2} \chi^{RPI\lambda_1\lambda_2} C_{(1/2)i_1(1/2)i_2}^{Ii} |i_1\rangle |i_2\rangle \\ &\quad \times \sqrt{\frac{2L+1}{2J+1}} C_{(1/2)\lambda_1(1/2)-\lambda_2}^{S\Lambda} C_{L0S\Lambda}^{J\Lambda} |{}^{2S+1}L_J\rangle. \end{aligned} \quad (3)$$

As an example, the  $\rho$ -like state which belongs to the chiral multiplet  $|(0,1) + (1,0); 11^{--}\rangle$  and  $|(1/2, 1/2)_b; 11^{--}\rangle$  can be represented as<sup>1</sup>

$$|(0,1) + (1,0); 11^{--}\rangle = \sqrt{\frac{2}{3}} |1; {}^3S_1\rangle + \sqrt{\frac{1}{3}} |1; {}^3D_1\rangle, \quad (4)$$

$$|(1/2, 1/2)_b; 11^{--}\rangle = \sqrt{\frac{1}{3}} |1; {}^3S_1\rangle - \sqrt{\frac{2}{3}} |1; {}^3D_1\rangle. \quad (5)$$

Since both, the chiral and  ${}^{2S+1}L_J$  representations are complete for two-particle systems with the quantum numbers  $I, J^{PC}$ , the angular momentum expansion is uniquely determined for each chiral state. Chiral symmetry imposes strong restrictions on the spin and angular momentum distribution of a system. The decomposition has been used in Ref. 4 to test the chiral symmetry breaking of the  $\rho$  meson in the infrared, and at the same time, to reconstruct its spin and orbital angular momentum content in terms of partial waves. This was achieved by using interpolators that transform according to  $|(0,1) + (1,0); 11^{--}\rangle$  and  $|(1/2, 1/2)_b; 11^{--}\rangle$ . If chiral symmetry were not broken there would be only two possible chiral states in the meson, while chiral symmetry breaking would imply a superposition of both. The obtained result in Ref. 4 indicates that the  $q\bar{q}$  component of the  $\rho$ -meson in the infrared is indeed a superposition of the  $|(0,1) + (1,0); 11^{--}\rangle$  and  $|(1/2, 1/2)_b; 11^{--}\rangle$  chiral states, and therefore chiral symmetry turns out to be broken. By using transformation (4) and (5) the partial wave content can be extracted, obtaining for the particular case of the  $\rho$  meson, a nearly pure  ${}^3S_1$  state.<sup>4</sup> This is an example of application, see also Refs. 1,5.

In this paper we will not discuss problems in which the chiral basis or its transformation can play a role as was done in Refs. 1,4 or 5, for instance. The problem we want to address here is more technical and related to the transformation (3) itself. The unitary transformation (3) was obtained in the instant form of relativistic quantum mechanics. In this work we investigate the corresponding expression one should use in the context of approaches that use light-front quantization<sup>8</sup> or

front-form relativistic quantum mechanics.<sup>7</sup> We pose the question whether the transformation (3) is identical in any other form<sup>6,7</sup> or if it is a special feature of those that use canonical spin, such as the instant- or the point-forms. The problem is not trivial since in relativistic composite systems the internal degrees of freedom transform among themselves nontrivially under rotations.<sup>7</sup> Relativity mixes spatial and temporal components, and as a consequence, one is not allowed to treat boosts and angular momentum separately in general. The election of a particular representation matters and in some cases some of the symmetry properties of the Poincaré group might not be manifest. The front form is of special interest, since rotations do not form a subgroup of the kinematical group and hence, rotational invariance is not manifest. On the other hand, front-form boosts form a subgroup of the Poincaré group, and as a result, the front-form Wigner rotation becomes the identity.<sup>7</sup>

In this article we will show that the unitary transformation derived in Ref. 1 in instant form is indeed identical in the front form of relativistic quantum mechanics. The argument resides in the fact that the *generalized Melosh rotation* that transforms front-form spins to helicity ones, becomes the identity when the mass goes to zero.<sup>10–12</sup>

## 2. Instant-form decomposition

Due to rotational and translational invariance in nonrelativistic quantum mechanics, the angular momentum coupling of two particles with individual spin and orbital angular momentum  $(\mathbf{s}_1, \mathbf{l}_1)$  and  $(\mathbf{s}_2, \mathbf{l}_2)$  to give a composite system of total spin and orbital angular momentum  $(\mathbf{S}, \mathbf{L})$  is easily realized by using the  $SU(2)$  Clebsch-Gordan coefficients. Relativity involves however, a change of representation in which the single-particle momenta and spins are replaced by an overall system momentum and internal angular momentum.<sup>7</sup> It is customary to use the Clebsch-Gordan coefficients of the Poincaré group.<sup>7</sup>

The kind of spin vector can be fully determined through the choice of a certain type of boost. Canonical boosts are rotationless. Spin vectors defined through canonical boosts have the advantage that in the center of momentum frame they transform under rotations in the same way as in nonrelativistic quantum mechanics, and therefore, for a composite system one can find a direct decomposition in terms of  $SU(2)$  Clebsch-Gordan coefficients. The reason is that in the canonical case the Wigner rotation associated with a pure rotation, turns out to be the rotation itself.<sup>7</sup> This does not hold in general. In the front form, rotational invariance is not manifest, and an angular momentum decomposition in terms of Clebsch-Gordan coefficients requires additional transformations.

Expression (2) can be achieved in a straightforward manner in non relativistic quantum mechanics, as well as in the instant form of dynamics or in any other form that uses canonical spin. The derivation of (2) can be found in Ref. 3. We will reproduce it here in a basis of eigenstates of the Poincaré group in order to be able to refer some steps that concern what will be exposed in the next section.

4 *María Gómez-Rocha*

We decompose the spin part of a two-particle state with total canonical angular momentum  $J$  and  $\hat{z}$ -component  $M$ , orbital angular momentum  $L$  and total spin  $s$ , in terms of quantum numbers of the constituents in the center of momentum frame ( $\mathbf{P} = \mathbf{0}$ ), where the relative momentum is expressed as  $\mathbf{k} = \mathbf{k}_1 = -\mathbf{k}_2$ ,

$$|[LS]|\mathbf{k}|J; \mathbf{0} M\rangle = \sum_{M_L M_S} \sum_{\sigma_1 \sigma_2} \int d\hat{\mathbf{k}} |\mathbf{k}\sigma_1 - \mathbf{k}\sigma_2\rangle C_{s_1 \sigma_1 s_2 \sigma_2}^{SM_S} Y_{LM_L}(\hat{\mathbf{k}}) C_{LM_L SM_S}^{JM}, \quad (6)$$

where  $|\mathbf{k}\sigma_1 - \mathbf{k}\sigma_2\rangle := |\mathbf{k}\sigma_1\rangle |-\mathbf{k}\sigma_2\rangle$ ,  $s_{1(2)}$  and  $\sigma_{1(2)}$  are respectively the individual canonical spins and their  $\hat{z}$ -projections.

Given a particular direction  $\hat{\mathbf{n}}$ , where the tensor product state can be written

$$\langle \hat{\mathbf{n}} | \mathbf{k}\sigma_1 - \mathbf{k}\sigma_2 \rangle := \psi_{s_1 \sigma_1}(\mathbf{k}) \psi_{s_2 \sigma_2}(-\mathbf{k}) \delta^2(\hat{\mathbf{k}} - \hat{\mathbf{n}}), \quad (7)$$

one can write

$$\begin{aligned} \psi_{JLSM}(\mathbf{k}) &:= \langle \hat{\mathbf{n}} | [LS] | \mathbf{k} | J; \mathbf{0} M \rangle \\ &= \sum_{M_L M_S} \sum_{\sigma_1 \sigma_2} \psi_{s_1 \sigma_1}(\mathbf{k}) \psi_{s_2 \sigma_2}(-\mathbf{k}) C_{s_1 \sigma_1 s_2 \sigma_2}^{SM_S} Y_{LM_L}(\hat{\mathbf{k}}) C_{LM_L SM_S}^{JM}. \end{aligned} \quad (8)$$

In order to express it in terms of helicities one needs to transform states with canonical spin to a basis of states with helicity spin. The unitary transformation that provides this is a Wigner rotation whose argument corresponds to the angle between the  $z$ -axis and the direction of motion  $\hat{\mathbf{k}} := \mathbf{k}/|\mathbf{k}|$

$$\psi_{s_1 \sigma_1}(\mathbf{k}) = \sum_{\lambda_1} D_{\lambda_1 \sigma_1}^{(s_1)}(\hat{\mathbf{k}}) \psi_{s_1 \lambda_1}(\mathbf{k}), \quad (9)$$

$$\psi_{s_2 \sigma_2}(-\mathbf{k}) = \sum_{\lambda_2} D_{-\lambda_2 \sigma_2}^{(s_2)}(\hat{\mathbf{k}}) \psi_{s_2 -\lambda_2}(\mathbf{k}). \quad (10)$$

Replacing in (6) one gets

$$\begin{aligned} \psi_{JLSM}(\mathbf{k}) &= \sum_{M_S M_L} \sum_{\sigma_1 \sigma_2} \sum_{\lambda_1 \lambda_2} D_{\lambda_1 \sigma_1}^{(s_1)}(\hat{\mathbf{k}}) \psi_{s_1 \lambda_1}(\mathbf{k}) D_{-\lambda_2 \sigma_2}^{(s_2)}(\hat{\mathbf{k}}) \psi_{s_2 -\lambda_2}(\mathbf{k}) \\ &\quad \times Y_{LM_L}(\hat{\mathbf{k}}) C_{s_1 \sigma_1 s_2 \sigma_2}^{SM_S} C_{LM_L SM_S}^{JM}. \end{aligned} \quad (11)$$

It is convenient to write the spherical harmonics in terms of Wigner  $D$ -functions<sup>a</sup>

$$Y_{LM_L}(\hat{\mathbf{k}}) = \sqrt{\frac{2L+1}{4\pi}} D_{0M_L}^L(\hat{\mathbf{k}}) \quad (12)$$

in such a way that one can make use of the relation for the product of Wigner  $D$ -functions with the same argument for axially symmetric systems,<sup>9</sup>

$$D_{m'_1 m_1}^{(j_1)}(\hat{\mathbf{w}}) D_{m'_2 m_2}^{(j_2)}(\hat{\mathbf{w}}) = \sum_j C_{j_1 m'_1 j_2 m'_2}^{jm'} D_{mm'}^{(j)}(\hat{\mathbf{w}}) C_{j_1 m_1 j_2 m_2}^{jm}, \quad (13)$$

<sup>a</sup>Our notation differs from Ref. 9 by a factor  $i^L$  in the definition of the phase of the spherical function.

with  $m = m_1 + m_2$ ,  $m' = m'_1 + m'_2$ , and  $\hat{\mathbf{w}}$  accounting for the Euler angles. This leads to

$$\begin{aligned} \psi_{JLSM}(\mathbf{k}) &= \sum_{\lambda_1 \lambda_2} \sqrt{\frac{2J+1}{4\pi}} D_{\Lambda M_J}^J(\hat{\mathbf{k}}) \psi_{s_1 \lambda_1}(\mathbf{k}) \psi_{s_2 - \lambda_2}(\mathbf{k}) \\ &\quad \times \sqrt{\frac{2L+1}{2J+1}} C_{s_1 \lambda_1 s_2 - \lambda_2}^{S\Lambda} C_{L 0 S \Lambda}^{J\Lambda}. \end{aligned} \quad (14)$$

The fact that the Wigner  $D$ -functions in (11) have the same argument, is a particular feature of the instant form, and it is restricted to the rest frame.<sup>7</sup>

It is easy to identify the needed matrix elements as

$$\psi_{JLSM}(\mathbf{k}) = \sum_{\lambda_1 \lambda_2} \psi_{JM \lambda_1 \lambda_2}(\mathbf{k}) \langle JM \lambda_1 \lambda_2 |^{2S+1} L_J M \rangle, \quad (15)$$

with

$$\psi_{JM \lambda_1 \lambda_2}(\mathbf{k}) := \sqrt{\frac{2J+1}{4\pi}} D_{\Lambda M_J}^J(\hat{\mathbf{k}}) \psi_{s_1 \lambda_1}(\mathbf{k}) \psi_{s_2 - \lambda_2}(\mathbf{k}) \quad (16)$$

and

$$\langle JM \lambda_1 \lambda_2 |^{2S+1} L_J M \rangle = \sqrt{\frac{2L+1}{2J+1}} C_{s_1 \lambda_1 s_2 - \lambda_2}^{S\Lambda} C_{L 0 S \Lambda}^{J\Lambda}. \quad (17)$$

This permits the translation from two-particle helicity states with total angular momentum  $J$ , to a one-particle state of overall orbital angular momentum  $L$  and intrinsic spin  $S$ . The connexion with chirality is immediately given by (1).

### 3. Front-Form Decomposition

Eq. (6) describes the angular momentum decomposition of a representation of canonical spin into a superposition of representations with canonical spin. Because on the front-form rotations do not form a subgroup of the kinematical group of the Poincaré group, the decomposition (6) is not feasible *a priori*. In order to analyze the coupling of two representations with individual spin to a superposition of representations with total spin for an arbitrary case in relativistic quantum mechanics, it is necessary to use a consistent expression of the Clebsch-Gordan coefficients of the Poincaré group.<sup>7</sup> Front-form angular momentum coupling is well known and it has been widely applied to hadron problems in front-form relativistic quantum mechanics. A relation of the type of (3), however, has not been established yet in the front form. This is the aim of the present section.

In the following we will use the normalization criteria and notation of Ref. 7. The light-front components of the four-momentum are defined by  $\tilde{\mathbf{p}} := (p^+ = p^0 + p^3, \mathbf{p}_\perp = (p^1, p^2))$ ,  $p^- = p^0 - p^3$ .  $|\tilde{\mathbf{p}}\mu\rangle_f$  represents a single particle state in a front-form basis (denoted by  $f$ ), with  $\hat{z}$ -spin projection  $\mu$ . The expression for the Clebsch-Gordan coefficients of the Poincaré group in the front form for an arbitrary

6 *María Gómez-Rocha*

frame is given by<sup>7</sup>

$$\begin{aligned}
 & {}_f \langle \tilde{\mathbf{p}}_1 \mu_1 \tilde{\mathbf{p}}_2 \mu_2 | [LS] |\mathbf{k}| J; \tilde{\mathbf{P}} M \rangle_f \\
 &= \delta(\tilde{\mathbf{P}} - \tilde{\mathbf{p}}_1 - \tilde{\mathbf{p}}_2) \frac{1}{|\mathbf{k}|^2} \delta(\mathbf{k}(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2) - \mathbf{k}) \left| \frac{\partial(\tilde{\mathbf{P}}, \mathbf{k})}{\partial(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2)} \right|^{1/2} \\
 &\times \sum_{\sigma_1 \sigma_2} D_{\mu_1 \sigma_1}^{(s_1)} [\mathbf{R}_{fc}(\mathbf{k}, m_1)] D_{\mu_2 \sigma_2}^{(s_2)} [\mathbf{R}_{fc}(-\mathbf{k}, m_2)] \\
 &\times Y_{ML}^L(\hat{\mathbf{k}}) C_{s_1 \sigma_1 s_2 \sigma_2}^{SM_S} C_{LM_L SM_S}^{JM}, \tag{18}
 \end{aligned}$$

where  ${}_f \langle \tilde{\mathbf{p}}_1 \mu_1 \tilde{\mathbf{p}}_2 \mu_2 |$  represents a tensor-product state of two particles with individual momenta  $\tilde{\mathbf{p}}_1$  and  $\tilde{\mathbf{p}}_2$  and spin  $\hat{z}$ -projections  $\mu_1$  and  $\mu_2$  respectively. The system of two particles moves with a total light-front momentum  $\tilde{\mathbf{P}}$  and the individual spins couple to give a total angular momentum  $J$  with orbital and spin contributions  $[LS]$  in the rest frame in the canonical form, and total angular momentum projection on the  $\hat{z}$ -direction,  $M$ . Finally,  $\mathbf{k} = \mathbf{k}_1 = -\mathbf{k}_2$  is used to denote the individual momenta in the rest frame in the canonical form, and  $m_1$  and  $m_2$  denote the individual constituent masses (they should not be confused with the spin projections, which appear in italics in equation (13)). The arguments of the Wigner  $D$ -functions are Melosh rotations which transform states with canonical spin to states with front-form spin and vice versa. Note that the rotation depends on the mass in general, producing a different effect on each constituent. Unless we are dealing with a system of identical constituent masses (e.g. the chiral case), we will not be able to use the properties of the  $D$ -function with the same argument as was done in the instant form.

The Clebsch-Gordan coefficient (18) is consistent with the normalization condition for single states

$${}_f \langle \tilde{\mathbf{p}}' \mu' | \tilde{\mathbf{p}} \mu \rangle_f = \delta(\tilde{\mathbf{p}} - \tilde{\mathbf{p}}') \delta_{\mu \mu'} \tag{19}$$

and for state vectors of overall momentum  $\tilde{\mathbf{P}}$

$$\begin{aligned}
 & {}_f \langle [L'S'] |\mathbf{k}'| J'; \tilde{\mathbf{P}}' M' | [LS] |\mathbf{k}| J; \tilde{\mathbf{P}} M \rangle_f \\
 &= \delta_{M'M} \delta_{j'j} \delta_{l'l} \delta_{s's} \delta(P'^+ - P^+) \delta^2(\mathbf{P}'_{\perp} - \mathbf{P}_{\perp}) \frac{1}{|\mathbf{k}|^2} \delta(\mathbf{k} - \mathbf{k}'). \tag{20}
 \end{aligned}$$

The problem now is to couple a state of total front-form angular momentum  $J$  and spin projection  $M$ ,  ${}_f \langle [LS] |\mathbf{k}| J; \tilde{\mathbf{P}} M \rangle_f$ , to a tensor-product state of two particles with individual spins described in terms of helicities  ${}_h \langle \mathbf{p}_1 \lambda_1 \mathbf{p}_2 \lambda_2 |$ .

Irreducible representations with different types of spin are related to each other through a unitary transformation.<sup>7</sup> The unitary transformation that relates helicity spin to front-form spin becomes the identity for massless particles.<sup>10–12</sup> This means:

$$| \tilde{\mathbf{p}}_1 \mu_1 \tilde{\mathbf{p}}_2 \mu_2 \rangle_f \xrightarrow{m \rightarrow 0} \sum_{\lambda_1 \lambda_2} | \tilde{\mathbf{p}}_1 \lambda_1 \tilde{\mathbf{p}}_2 \lambda_2 \rangle_h \delta_{\lambda_1 \mu_1} \delta_{\lambda_2 \mu_2} \tag{21}$$

where the subindex  $h$  labels helicity states. Front-form spins and helicity spins coincide in the chiral limit, and one is allowed to make use of them without distinction.

Replacing (21) in (18), one obtains the Clebsch-Gordan coefficient that couples two-particle helicity states to an overall state of the front-form basis,

$$\begin{aligned}
 & {}_h \langle \tilde{\mathbf{p}}_1 \lambda_1 \tilde{\mathbf{p}}_2 \lambda_2 | [LS] |\mathbf{k}; J; \tilde{\mathbf{P}} M \rangle_f \\
 &= \delta(\tilde{\mathbf{P}} - \tilde{\mathbf{p}}_1 - \tilde{\mathbf{p}}_2) \frac{1}{|\mathbf{k}|^2} \delta(\mathbf{k}(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2) - \mathbf{k}) \left| \frac{\partial(\tilde{\mathbf{P}}, \mathbf{k})}{\partial(\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2)} \right|^{1/2} \\
 &\times \sum_{\sigma_1 \sigma_2} D_{\lambda_1 \sigma_1}^{(s_1)} [R_{hc}(\hat{\mathbf{k}})] D_{\lambda_2 \sigma_2}^{(s_2)} [R_{hc}(-\hat{\mathbf{k}})] \\
 &\times Y_{M_L}^L(\hat{\mathbf{k}}) C_{s_1 \sigma_1 s_2 \sigma_2}^{SM_S} C_{LM_L SM_S}^{JM}. \tag{22}
 \end{aligned}$$

Now the Melosh rotations  $D_{\lambda_1 \sigma_1}^{(s_1)} [R_{hc}(\hat{\mathbf{k}})]$  and  $D_{\lambda_2 \sigma_2}^{(s_2)} [R_{hc}(-\hat{\mathbf{k}})]$  are equivalent to the Wigner rotations and they only depend on the direction of  $\mathbf{k}$ . They have exactly the same significance as in (11): they transform canonical spins into helicity spins. We are in the position to write the expression for the state in which we are interested

$$\begin{aligned}
 | [LS] |\mathbf{k}; J; \tilde{\mathbf{P}} M \rangle_f &= \sum_{\lambda_1 \lambda_2} \int d^3 \tilde{p}_1 d^3 \tilde{p}_2 |\tilde{\mathbf{p}}_1 \lambda_1 \tilde{\mathbf{p}}_2 \lambda_2 \rangle_h \\
 &\times {}_h \langle \tilde{\mathbf{p}}_1 \lambda_1 \tilde{\mathbf{p}}_2 \lambda_2 | [LS] |\mathbf{k}; J; \tilde{\mathbf{P}} M \rangle_f, \tag{23}
 \end{aligned}$$

where  $\mathbf{1} = \sum \int d^3 \tilde{p}_1 d^3 \tilde{p}_2 |\tilde{\mathbf{p}}_1 \lambda_1 \tilde{\mathbf{p}}_2 \lambda_2 \rangle_h {}_h \langle \tilde{\mathbf{p}}_1 \lambda_1 \tilde{\mathbf{p}}_2 \lambda_2 |$  has been introduced.

Reexpressing it in terms of  $\tilde{\mathbf{P}}$  and  $\mathbf{k}$  and setting the center of momentum frame,  $\tilde{\mathbf{P}} = \tilde{\mathbf{0}} := (2p^0, 0, 0, 0)$ ,

$$\begin{aligned}
 | [LS] |\mathbf{k}; J; \tilde{\mathbf{0}} M \rangle_f &= \sum_{\lambda_1 \lambda_2} \sum_{\sigma_1 \sigma_2} \int d\hat{\mathbf{k}} |\mathbf{k} \lambda_1 - \mathbf{k} \lambda_2 \rangle \\
 &\times D_{\lambda_1 \sigma_1}^{(s_1)} [R_{hc}(\hat{\mathbf{k}})] D_{\lambda_2 \sigma_2}^{(s_2)} [R_{hc}(-\hat{\mathbf{k}})] \\
 &\times Y_{M_L}^L(\hat{\mathbf{k}}) C_{s_1 \sigma_1 s_2 \sigma_2}^{SM_S} C_{LM_L SM_S}^{JM}. \tag{24}
 \end{aligned}$$

Setting now a particular direction of motion  $\hat{\mathbf{n}}$ , the integral over  $d\hat{\mathbf{k}}$  can be carried out by means of

$$\langle \hat{\mathbf{n}} | \mathbf{k} \lambda_1 - \mathbf{k} \lambda_2 \rangle := \psi_{s_1 \lambda_1}(\mathbf{k}) \psi_{s_2 \lambda_2}(-\mathbf{k}) \delta(\hat{\mathbf{k}} - \hat{\mathbf{n}}). \tag{25}$$

And we have

$$\begin{aligned}
 \psi_{JLSM}(\mathbf{k}) &:= \langle \mathbf{n} | [LS] |\mathbf{k}; J; \tilde{\mathbf{0}} M \rangle_f \\
 &= \sum_{\lambda_1 \lambda_2} \sum_{\sigma_1 \sigma_2} \psi_{s_1 \lambda_1}(\mathbf{k}) \psi_{s_2 -\lambda_2}(\mathbf{k}) D_{\lambda_1 \sigma_1}^{(s_1)} [R_{hc}(\hat{\mathbf{k}})] D_{-\lambda_2 \sigma_2}^{(s_2)} [R_{hc}(\hat{\mathbf{k}})] \\
 &\times Y_{M_L}^L(\hat{\mathbf{k}}) C_{s_1 \sigma_1 s_2 \sigma_2}^{SM_S} C_{LM_L SM_S}^{JM}. \tag{26}
 \end{aligned}$$

Proceeding in the same way as in the previous section in the combination of the spherical harmonic and the Wigner  $D$ -functions one obtains

$$\psi_{JLSM}(\mathbf{k}) = \sum_{\lambda_1 \lambda_2} \psi_{JM \lambda_1 \lambda_2}(\mathbf{k}) \langle JM \lambda_1 \lambda_2 |^{2S+1} L_J M \rangle \tag{27}$$

8 *María Gómez-Rocha*

being again

$$\psi_{JM\lambda_1\lambda_2}(\mathbf{k}) := \sqrt{\frac{2J+1}{4\pi}} D_{\Lambda MJ}^J(\hat{\mathbf{k}}) \psi_{s_1\lambda_1}(\mathbf{k}) \psi_{s_2-\lambda_2}(\mathbf{k}), \quad (28)$$

and

$$\langle JM\lambda_1\lambda_2 |^{2S+1} L_J M \rangle = \sqrt{\frac{2L+1}{2J+1}} C_{s_1\lambda_1 s_2-\lambda_2}^{S\Lambda} C_{L0S\Lambda}^{J\Lambda}. \quad (29)$$

Having found (29), the validity of decomposition (3) is demonstrated.

Unlike in the instant form, the combination of the Wigner  $D$ -functions would not have been possible if we had considered particles of different masses. Only in the chiral limit, or for equal masses, the eigenstates in the rest frame transform in the same way as in nonrelativistic quantum mechanics. Note that in general the coupling (18) involves rotations that depend on the masses, namely  $D_{\mu_1\sigma_1}^{(s_1)}[\mathbf{R}_{fc}(\mathbf{k}, m_1)]$  and  $D_{\mu_2\sigma_2}^{(s_2)}[\mathbf{R}_{fc}(-\mathbf{k}, m_2)]$ . This would have prevented the application of (13), since the  $D$ -functions would not have the same arguments, and the dependence on the masses would have entered the decomposition, making it impossible to write (26) in the form of a product of (28) and (29). Moreover, a further rotation would have been necessary in order to transform front-form spins to helicity spins, which in the chiral limit turns out to be trivial by means of (21).

The result is that decomposition (3) can be therefore used to expand chiral states as a superposition of vectors of the  $\{I;^{2S+1} L_J\}$ -basis within a front-form framework. It can be expressed as

$$\begin{aligned} |R; IJ^{PC}\rangle_f &= \sum_{LS} \sum_{\lambda_1\lambda_2} \sum_{i_1i_2} \chi_{\lambda_1\lambda_2}^{RPI} C_{(1/2)i_1(1/2)i_2}^{Ii} |i_1\rangle |i_2\rangle \\ &\times \sqrt{\frac{2L+1}{2J+1}} C_{(1/2)\lambda_1(1/2)-\lambda_2}^{S\Lambda} C_{L0S\Lambda}^{J\Lambda} |^{2S+1} L_J\rangle_f. \end{aligned} \quad (30)$$

#### 4. Summary

We have derived the unitary transformation that relates the  $q\bar{q}$  chiral basis to the  $\{I;^{2S+1} L_J\}$ -basis in a front-form framework. The result turns out to be the same as in instant form.<sup>1</sup>

Spin vectors belonging to different representations can be related through a unitary transformation.<sup>7</sup> We have used the feature of the generalized Melosh rotation that relates helicity and front-form spins, which becomes the identity for massless particles. This has made it possible to find a simple expression in terms of  $SU(2)$  Clebsch-Gordan coefficients, by starting from the most general case of the Clebsch-Gordan coefficients of the Poincaré group. The limit  $m \rightarrow 0$  eliminates the mass-dependence in the Wigner  $D$ -functions making it possible to express the product of  $D$ -functions with the same argument through a Clebsch-Gordan series for axially symmetric systems.

To better understand the significance of this calculation let us recall that the purpose of the Clebsch-Gordan coefficients of the Poincaré group is to convert any



kind of spin to canonical spin in the rest frame, in such a way that they can be added using  $SU(2)$  Clebsch-Gordan coefficients.<sup>7</sup> In the system rest frame, the only feature that distinguishes front-form spins from canonical ones is the fact that front-form spins are characterized for being invariant under front-form boosts. This difference is accounted by the Melosh rotation (cf. (18)).

As a last remark, let us also mention that it would have been possible to develop such a decomposition for any type of spin. The Clebsch-Gordan coefficients of the Poincaré group for an arbitrary form are given in Ref. 7. Proceeding in an analogous way as before, it is possible to see that again the Wigner  $D$ -functions do not have the same argument, and it is not possible to bring them together to an overall rotation by means of  $SU(2)$  Clebsch-Gordan coefficients. Only in the chiral limit the rotations are again the same. But unlike in the front form, an additional transformation on such arbitrary spins into helicity spins is necessary to make the relation to chirality possible.

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